Some Perspectives on Patterns in Middle School Mathematics Curricula

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Abstract: In this paper we analyze the development of numerical and geometrically defined patterns in a variety of middle school curricula and pay special attention to the mathematical treatment of the relevant concepts. Part of our analysis involves the consideration of important factors influencing different curricular approaches to the study of patterns in the middle grades. In particular, we review U.S. standards and state framework documents and highlight the interplay between such documents and published curriculum materials. We provide examples that illustrate a need for rethinking how the study of patterns and generalizations are presented in middle grade mathematics curricula, and discuss implications to the mathematical preparation of middle grade teachers.

1. Introduction

Patterns and relationships are at the core of mathematics, and are accordingly prominent throughout school mathematics curriculum. Mathematicians attempt to understand fundamental structures by searching for patterns and relationships within classes of examples and collections of data. Their investigations involve insightful questions and conjectures in unison with creative thinking and problem-solving strategies, and it is especially crucial for all students of mathematics to comprehend and embrace these habits of mind.

Lynn Steen, past president of the Mathematical Association of America, 1985-86, echoes similar sentiments in the book, *On the Shoulders of Giants: New Approaches to Numeracy* (1990):

> What humans do with the language of mathematics is to describe patterns. Mathematics is an exploratory science that seeks to understand every kind of pattern–patterns that occur in nature, patterns invented by the human mind, and even patterns created by other patterns. To grow mathematically, children must be exposed to a rich variety of patterns appropriate to their own lives through which they can see variety, regularity and interconnection. (p. 8)

Given the significance of patterns and relationships in the discipline of mathematics, it is interesting to understand how they are introduced and studied in school mathematics curricula, and to determine if their employment is mathematically meaningful and supports student-learning objectives. Our work will focus primarily on the mathematical treatment of numerically and geometrically defined patterns in the middle grade curriculum in regards to the teaching and learning of beginning algebraic concepts and procedures. Our analysis will involve a discussion of the college level mathematics underlying these ideas, and some implications to the mathematical preparation of middle grade teachers. An important aspect of our efforts will

be the examination of national recommendations and assessments that influence curricular content.

2. Establishing the importance of numerical and geometric pattern exploration

In an effort to understand the dynamics of why and how numerically and geometrically described patterns are included in middle grade curricula, it is essential to consider national policy recommendations on curricular content. National and state standards documents outline the importance of studying patterns and sequences in middle grades mathematics in the U.S. These documents both describe ways in which the study of pattern and sequence concepts can be ideally presented to students, as well as reasons for the inclusion of this content in these documents.

U.S. mathematics textbooks provide not only the problems to be presented to and worked on by students, but the authors also offer insights into why students should study pattern and sequence concepts. In this section we report on our examination of selected standards documents and textbooks in connection with pattern concepts and problems, as well as reasons for the inclusion of such problems in certain textbooks and curricular documents.

2.1 National and state standard documents

The National Council of Teachers of Mathematics' (NCTM) standards documents were written to provide guidance to developers of state and district standards documents, as well as published textbooks and materials. "This document is intended to set forth a comprehensive and coherent set of goals for mathematics for all students from pre-kindergarten through grade 12 that will orient curricular, teaching, and assessment efforts during the next decades" (NCTM, 2000, p. 6).

Our analysis of the collection of NCTM standards documents regarding mathematics content involving the study of patterns and functions in the middle grades vividly shows that these topics are viewed as mathematically important and developmentally appropriate for middle grade students. The following quote is one example supporting this notion:

> "In grades 6-8 all students should understand patterns, relations, and functions: represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules" (NCTM, 2000, p. 222).

Value judgments regarding the mathematics that students should learn are presented throughout the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) standards for Grades 5-8, where increased attention on patterns and relationships is recommended:

> In grades 5-8, the mathematics curriculum should include explorations of patterns and functions so that students can—
> - describe, extend, analyze, and create a wide variety of patterns;
> - describe and represent relationships with tables, graphs, and rules;
> - analyze functional relationships to explain how a change in one quantity results in a change in another;
> - use patterns and functions to represent and solve problems. (p. 98)

Students' work with patterns and functions in the elementary grades to build a foundation for future studies of patterns, functions, and generalizations (NCTM, 1989). Although more emphasis is placed on functions in the middle grades, students continue to use their understanding of patterns and generalizations established in elementary school to explore the mathematical relationships present in the middle school curriculum.

The NCTM standards documents provide general grade band (K-2, 3-5, 6-8, and 9-12) learning expectations concerning patterns and generalizations, and in turn, state policy makers, district and school administrators, teachers, publishers, and curriculum developers interpret and modify them to meet their own particular needs. Because of these diverse decision making forces, there is great variability in grade level learning expectations from state to state (Reys et al., 2006; Reys & Lappan, 2007). Below is one example from the NCTM standards documents, *Principles and Standards for School Mathematics* (NCTM, 2000), from which state and local policy makers make judgments for students concerning learning objectives. It is easy to see that these general guidelines are open to multiple interpretations:

> In grades 3-5, they [students] can begin to use variables and algebraic expressions as they describe and extend patterns. By the end of secondary school, they should be comfortable using the notation of functions to describe relationships. (NCTM, 2000, p. 38)

As noted above, NCTM standards have influenced the development of individual state documents. As such, these state standards contain a similar amount of openness for interpretation as those found in the NCTM documents. One characteristic example is South Dakota's Mathematics Content Standards, in which the authors acknowledge the influence of the *PSSM* (NCTM, 2000), among other documents (South Dakota Department of Education, 2004). The authors of the chapter *Guiding Principles and Key Components of an Effective Mathematics Program*, in the California state document *Mathematics Framework for California Public Schools, Kindergarten through Grade Twelve* (2005), also note the importance of patterns in the study of mathematics, and argue that "looking for patterns" is an important strategy in solving any particular problem.

While state's documents provide various reasons for the study of patterns. Some, such as South Dakota, also provide examples of what pattern problems should look like. One such example from the South Dakota 6-8 Standards is as follows:

Complete the table and write an algebraic expression for the given table.

x	1	2	3	4
y	7	14	21	?

2.2 Textbook examples, rationales, and contexts

In our examination of middle school mathematics textbooks, we found varying rationales and examples of pattern concept problems. Rationales that are tied to the study of algebra are numerous, however, some authors also justify the study of patterns in relation to their use in other disciplines such as science. For example, authors of one textbook note that, "Marine biologists use numerical patterns when determining the times for high tides and low tides" (Charles, Dossey, Leinwand, et al., 2002, p. 102). Other authors have couched the study of patterns in the larger framework of general studies in mathematics, "Patterns are at the heart of

mathematics, and you can find patterns by looking at shapes, numbers, and many other things" (National Center for Research in Mathematical Sciences, 1998, p. 6).

While some rationales are related to mathematics that may subsequently appear in the curriculum at an accessible level, one textbook relates pattern concepts to complex areas of mathematics not accessible to many 6-12 grade students, such as fractal geometry and complex population models (Charles, et al., 2002). In each of these examples, the authors attempt to frame the study of pattern concepts. However, it is clear that the authors have multiple perspectives of how patterns should be taught, the overall importance of patterns, and their interpretations of the standards related to studying patterns.

3. Numerical and geometric patterns in middle grade mathematics curricula

The middle grades are a dynamic developmental period for children (Hart, 1998; Lappan & Ferrini-Mundy, 1993; Southern Regional Education Board, 2003), and a time when they are introduced to a variety of new mathematical concepts, procedures, and reasoning strategies, several of which fall into the domain of algebra. As middle grade students begin to develop the ideas of variable and function, there is gradual transformation from describing rules using language to describing rules using symbols, and this is a particularly critical transition period. Curricular materials take varied approaches to guide students through this difficult learning phase, and it is our intention to limit the focus of our analysis to segments of approaches that involve recognition and algebraic representations of numerical and geometric pattern progressions.

3.1 Content analysis focus

The curricula we have investigated have many different examples and exercises involving the description of such patterns, and our objectives include critically analyzing the nature, purpose, and mathematical soundness of several of these problems. With respect to a meaning for the qualitative property, *mathematical soundness,* we have adopted (with slight modifications for our purposes) some defining characteristics utilized by Hyman Bass (2005) in his Brookings Institution assessment analysis of selected Algebra items from the NAEP instrument. In particular, we will evaluate the mathematical soundness of a common pattern-type problem in terms of the following criteria:

(a). A problem should be clearly and precisely formulated at a level accessible to the intended grade, and free of non-purposeful ambiguity.
(b). A problem should involve substantial mathematical knowledge and/or skills.
(c). If a problem is contextualized, it should have a sensible context that is taken seriously and one that does not obscure the intended mathematical focus of the item.

These criteria might not characterize the meaning of mathematical soundness for all readers, however they do constitute a reasonable and coherent basis for our qualitative analysis. To read Bass' (2005) analysis in its entirety, as well as the companion analyses of Roger Howe and James Milgram (from their panel presentation entitled: *Analysis of NAEP Items Classified Under the Algebra and Functions Content Strand*), search the archives on the Brookings Institution website, (www.brookingsinstitution.org).

3.2 Number patterns: linear sequences

Now that we have described the extent of our analysis, let us first consider several typical numerical pattern-type problems appearing in contemporary curricula. A common such problem encountered in middle grade curricula is one that involves identifying a pattern associated with a finite list of positive integers. For example, familiar number lists that have appeared in middle grade curricula are:

a) 3, 5, 7,…
b) 2, 4, 8, 16,…
c) 1,1, 2, 3, 5, 8, 13,…

Tasks associated with these lists are often prompted with questions of the following type:

1. Given a list with three or four numbers, what is the 6th term of the list, the 8th term, the 11th term, etc.?
2. Make a table displaying the first 20 terms of the given list.
3. Write a rule for finding the n^{th} term of the given list, and show (provide some argument) that the rule works for all positive integers n (i.e., find a justifiable rule).
4. Use your rule to determine if the number 79 appears in the list, and if it does, what term would 79 be in the list?

Although the given lists only display a finite set of numbers, the notation used and the questions posed indicate that each list is assumed to continue on indefinitely governed by an inherent structure, and that the remaining terms can be determined by some specific rule related to that structure. In particular, the nature of these problems implies that each given list is understood to be a *sequence*, i.e., a function defined from the positive integers into any nonempty set. Since, by definition, the domain of a sequence f is the positive integers, it is standard practice to represent the function f in the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots,$$

where, $f(n) = a_n$ for each positive integer n.

When a student first encounters a problem of the type described above, their focus (which is influenced by teacher and curriculum) is on finding the *expected* solution, rather than thinking about if the given problem makes mathematical sense, and whether the provided information actually leads to the anticipated result. A usual opening strategy is for them to calculate changes between the successive terms of the given sequence and then attempt to use those changes to describe subsequent terms.

For example, in the list 3, 5, 7,…, a student might proceed by observing that each succeeding term is two more than the preceding term, and thus the fourth term should be 9, the fifth term 11, and the sequence can be represented in general by the rules, $a_1 = 3$ and $a_n = a_{n-1} + 2$. This kind of recursive reasoning and representation approach is a frequent strategy employed by children when initially working with elementary pattern problems, rather than finding a closed form formula to represent the sequence, i.e., finding an explicit rule f expressing the n^{th} member of the sequence in terms of the positive integer n (in this case, $f(n) = 2n + 1$).

Although it is common for curricular materials, teachers, and students to declare that 9 is the unique answer to the question:

"What is the fourth term in the sequence 3, 5, 7, …?",

it is in fact an ill-posed question that does not have a unique answer. For example, one student might say the answer is 11, since they thought the list was of all odd prime numbers, whereas another student might say the next term is 3, since they thought the sequence was periodic and just repeated the first three terms forever. These are just two conceivable answers derived from the infinitely many possible sequences f (functions f defined on the positive integers), where $f(1) = 3$, $f(2) = 5$, and $f(3) = 7$. Without knowing more about the defining properties of the sequence f or some underlying structure that the sequence emerged from, it is simply speculation (or in the vernacular of children, "a guessing game") to determine $f(n)$ for any $n \geq 3$, rather than a precise mathematical deduction. In summary, we would classify this problem as one that is not mathematically sound, since it fails to meet criterion (a), and to some extent criterion (b) (Bass, 2005).

Since a principal purpose of these sequence problems (as stated in standards documents and curriculum learning expectation guides) is introducing students to fundamental algebraic concepts (such as variables and functions), and algebraic procedures (such as solving equations), it is crucial to present the problems as coherent well-defined mathematical statements that support the desired learning goals. For example, one uncomplicated reformulation of the previous problem into more precise mathematical language, and one that meets the intended algebraic learning objectives, is as follows:

Restated problem: Find a rule for a sequence whose first three terms are 3, 5, 7, and then use that rule to determine the fourth, fifth and sixth terms of the sequence. Another rephrasing of the problem is to find a rule that produces the below table, and then use that rule to expand the table to one that contains positions 4, 5, 6, etc.

Position in the list	Term in the list
1	3
2	5
3	7

As was remarked earlier, the recursive rule $a_1 = 3$ and $a_n = a_{n-1} + 2$ (where the subscripts correspond to positions in the list) provides a representation of the list, and certainly is a solution to the problem. However, an alternate solution would be to find a closed form rule f that describes the list (such as a polynomial function), i.e., a rule that explicitly relates the position in the list (first, second, third) to the terms in the list (three, five, seven). When students are asked to find an explicit rule representing this simplistic pattern, a common approach for them is to generalize the arithmetic calculations

$$3 = (2 \bullet 1) + 1,\ 5 = (2 \bullet 2) + 1,\ 7 = (2 \bullet 3) + 1,$$

to conclude that the rule $f(n) = 2n + 1$ produces the list.

At this point, two important comments are appropriate. First, even though the discovered recursive rule and the closed form rule both produce the given table, that does not automatically guarantee that they will generate the same extended tables. In this case, the rules do generate the same sequences, and how this conclusion could be justified should be part of the classroom

discussion. While it is not necessary to formally describe the Principle of Mathematical Induction at the middle grade level, this kind of problem provides a good opportunity to introduce the ideas behind a proof by induction. Hence, it is most important that middle grade teachers have a firm understanding of inductive proof, and how it is utilized in varied contexts.

Second, although the above closed form rule describes the given list, and can be used to extend it indefinitely, it is not the only closed form rule that has these properties. In particular there are infinitely many functions g defined from the positive integers into the integers having the values, $g(1) = 3$, $g(2) = 5$, and $g(3) = 7$, and moreover there are infinitely many polynomial functions with these given values. (In accordance with the standard convention, we will use the notation $g(x)$ to represent the polynomial, and g to represent its induced polynomial function). The polynomial function g induced by the cubic polynomial

$$g(x) = (2x + 1) + (x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 13x - 5$$

has the property that $g(1) = 3$, $g(2) = 5$, and $g(3) = 7$. However, $f(4) = 9$ and $g(4) = 15$, and thus even though f and g produce the given table, they do not generate the same sequences (i.e., their extended tables are not the same).

More generally, a polynomial function g induced by the polynomial $g(x)$ (for our purposes having rational number coefficients) satisfies $g(1) = 3$, $g(2) = 5$, and $g(3) = 7$ if and only if the polynomial $g(x)$ is given by

$$g(x) = (2x + 1) + (x - 1)(x - 2)(x - 3)h(x),$$

where $h(x)$ is some polynomial with rational coefficients. Although this observation and its proof are too sophisticated for a beginning algebra student, it is certainly content that is both appropriate and useful for middle grade and high school teachers. Not only does this result provide teachers with a method for constructing infinitely many polynomials having a specified finite set of values from a given polynomial with those same specified values, but also its proof is a nice application of the Factor Theorem.

In particular, if the polynomial function g induced by the polynomial $g(x)$ satisfies $g(1) = 3$, $g(2) = 5$, and $g(3) = 7$, then the polynomial function k induced by the polynomial $k(x) = g(x) - (2x+1)$ satisfies $k(1) = 0$, $k(2) = 0$, and $k(3) = 0$. Thus, by the Factor Theorem,

$$k(x) = (x - 1)(x - 2)(x - 3)h(x),$$

where $h(x)$ is some polynomial with rational coefficients, and so

$$g(x) = (2x + 1) + (x - 1)(x - 2)(x - 3)h(x).$$

While middle grade teachers would not give the above formal argument in their classrooms, their knowledge of such mathematics would assist them in teaching related ideas that are appropriate for the middle grade classroom. The goal of providing prospective teachers with a mathematics preparation that develops a deep understanding of the mathematics they will teach is a key element in improving student achievement, according to the Glenn Commission Report (National Commission on Mathematics and Science Teaching for the 21st Century, 2000), and is a major recommendation in the Mathematics Education of Teachers Report (American Mathematical Society, 2001).

In our analysis of example (a), we offered a restatement for that problem that transformed it into a well-defined mathematical problem, however an alternate way to present pattern-type problems, and one that authentically reflects the way mathematicians study patterns, is to create problems where these or similar sequences emerge from a structured mathematical or "real world" context. For example, consider the follow question:

Question: What positive integers can arise as the sum of four consecutive positive integers? List the five smallest sums that are possible in increasing magnitude, and find a rule for the n^{th} term of your list.

This exercise has similar learning goals to the other problems we have considered (i.e., recognize a pattern in a list of numbers and represent it by a rule), but it significantly differs from them in that there is a unique (infinite) sequence and corresponding rule that emerges from the mathematical structure of the problem. In particular, consider the following solution:

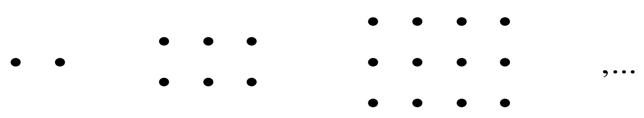
$$(1+2+3+4)=10,\ (2+3+4+5)=14,\ 18,\ 22,\ 26,\ldots,n+(n+1)+(n+2)+(n+3)=4n+6$$

This sort of problem affords children the opportunity to recognize and represent patterns as they occur in an actual mathematical setting, and in a manner that is consistent with mathematical discovery.

It is vital for middle and secondary mathematics teachers to have a mathematical preparation that provides them with a firm understanding of these and other important linear algebra concepts, and thus with a knowledge base that would assist them and their students in making important mathematical deductions and connections.

Next we consider another common type of problem framed within a geometric context. In fact, R. James Milgram (2006) critiques this problem in his unpublished preprint, *Pattern Recognition Problems in K-12*, and notes that it appeared, as stated, in the original proposed list of problems developed for President Clinton's planned national test in mathematics. (This problem was later revised based on suggestions made by Milgram and R. Schoen).

Problem: Consider the sequence of dot diagrams:



Assume that the number of dots added at each step is more than the number added in the previous step. How many dots in the 20^{th} term?

The solution to the problem, according to the test developers, is that 20x21=420 dots appear in the 20^{th} term. However, for the same reasons discussed in Section 3.2 concerning the number sequence problem 3, 5, 7, …, the three dot diagrams, plus the condition on the number of dots in subsequent diagrams, do not uniquely lead to the expected solution. Moreover, the dot diagrams are not especially germane to the problem as stated, since they do not provide any inherent structure that precisely defines the sequence, and could just have well been replaced by the number list 2, 6, 12. Hence, we would say this problem and others like it are not mathematically sound, since they have the same deficiencies as the number list problems that we previously analyzed.

Although the dot diagram problem as stated does not provide the necessary information to

uniquely determine the number of dots in the 20th term, Milgram (2006) does indicate that the conditions of the problem do lead to a lower bound for the number of dots in the 20th term, and in general the n^{th} term. He notes that the third term contains 6 more dots than the second term, so the fourth term must contain at least 7 more dots than the third term, and so on. He then concludes that the lower bound for the n^{th} term is given by the expression:

$$\frac{(n+3)(n+4)}{2} - 9, \ n \geq 3,$$

and so the lower bound for the 20th term, i.e., $n = 20$, is 267 dots. Using the formula for the sum of an arithmetic progression, one can arrive at the lower bound expression obtained by Milgram in the following manner:

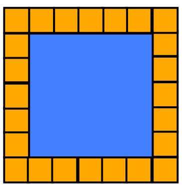
Finding the lower bound: Since the third term of the given dot sequence contains 6 more dots than the second term, i.e., 12 dots, the fourth term contains at least 12 + 7 dots, the fifth term contains at least 12 + 7 + 8 dots, and thus the n^{th} term contains at least 12 + 7 + 8 +…+ (n + 3) dots. Finally, notice that

$$12 + 7 + 8 + \ldots + (n+3) = [1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \ldots + (n+3)] - 9 = \frac{(n+4)(n+3)}{2} - 9.$$

A key point that Milgram (2006) makes in his article is that "pattern recognition", i.e., determining the next term or the n^{th} term in a list that lacks a given or derivable formation rule, is not meaningful mathematics, whereas, "pattern analysis", i.e., analyzing patterns that are defined by some specified rule or embedded in a context where a rule can be deduced, is sound mathematics that can lead to other significant mathematical ideas and connections. To help clarify his comments concerning "pattern analysis", he discusses how the analysis of a pattern, that arose in the area of topology, helped lead to the discovery of some deep results in that area.

Several of the contemporary curricula we examined do contain mathematically sound geometrically structured pattern problems that are middle grades appropriate and meet the desired learning objectives (i.e., recognize a pattern in a list of numbers and represent it by a justifiable rule) in a mathematically meaningful manner. We conclude this section with a representative example of such problems.

Problem: Tiling pools (Lappan et al., 2006, p. 6). In-ground pools are often surrounded by borders of tiles. The Custom pool Company gets orders for square pools of different sizes. For example, the pool below has side lengths of 5 feet and is surrounded by square border tiles. All Custom border tiles measure 1 foot on each side. A total of 24 tiles are needed for this particular pool.



How many border tiles do you need to surround a square pool (all borders are 1 tile wide)?

In order to calculate the number of tiles needed for a project, the Custom Pool manager wants an equation relating the number of border tiles to the size of the pool.

A. 1. Write an expression for the number of border tiles N based on the side length s of a square pool.
   2. Write a different but equivalent expression for the number of tiles N needed to surround the pool.
   3. Explain why your two expressions for the number of border tiles are equivalent.

B. 1. Use each expression in Question A to write an equation for the number of border tiles N. Make a table and a graph for each equation.
   2. Based on the table and graph, are the two expressions for the number of border tiles in Question A equivalent? Explain.

C. Is the relationship between the side length of the pool and the number of border tiles linear, exponential, quadratic, or none of these? Explain.

4. Concluding remarks

In our analysis of those problems that we found to lack mathematical soundness, we offered suggestions for improvement or gave alternate well-posed problems possessing the same learning goals. In addition, part of our analysis illuminated important mathematical content underlying these pattern type problems, and thus had implications to the mathematical preparation of middle grade teachers. Finally, it is vital to note that our work is not meant to be interpreted in a manner that suggests pattern type problems have little value in the pre-algebra/algebra curriculum, but rather, that with thoughtful modifications these kind of problems are an important part of the pre-algebra/algebra curriculum.

5. Notes

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